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INVERSE PROBLEM FOR A COUPLED PARABOLIC SYSTEM WITH DISCONTINUOUS CONDUCTIVITIES: ONE-DIMENSIONAL CASE.

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Abstract. We study the inverse problem of the simultaneous identification of two discontinuous diffusion coefficients for a one-dimensional coupled parabolic system with the observation of only one component. The stability result for the diffusion coefficients is obtained by a Carleman-type estimate. Results from numerical experiments in the one-dimensional case are reported, suggesting that the method makes possible to recover discontinuous diffusion coefficients.

1. **Introduction.** Let $\Omega =]0,1[$, $T>0$. We consider the following linear parabolic system:

$$\left\{\begin{array}{rcll} \partial_t u_1 - \partial_x(c_1\partial_x)u_1 & = & a_{11}u_1 + a_{12}u_2 & \text{in Q},\\ \partial_t u_2 - \partial_x(c_2\partial_x)u_2 & = & a_{21}u_1 + a_{22}u_2 + h & \text{in Q},\\ u_j(t,x) & = & 0 & \text{on } \Sigma,\ j=1,2,\\ u_j(0,x) & = & u_{0,j}(x) & \text{in } \Omega,\ j=1,2, \end{array}\right. \tag{1}$$

where we set $Q=\Omega\times(0,T)$, $\Sigma=\partial\Omega\times(0,T)$. For any $h\in L^2(Q)$, $a_{j,k}(t,x)\in L^\infty(Q)$, $1\le j,k\le 2$, $u_{0,1},u_{0,2}\in L^2(\Omega)$, $c_1,c_2\in L^\infty(\Omega)$ such that $c_1,c_2\ge\delta>0$, there exists a unique solution in $C^0([0,T];(H_0^1(\Omega)^2))$ to the linear system (1), (e.g., [12, Chap. 7], [14, Chap. 3]). We denote it by $\mathbf{U}(\mathbf{u_0},h;\mathbf{c}) = (U_1(\mathbf{u_0},h;\mathbf{c}),U_2(\mathbf{u_0},h;\mathbf{c}))$, with $\mathbf{c}=(c_1,c_2)$, $\mathbf{u_0}=(u_{0,1},u_{0,2})$.

Let us introduce the following inverse problem. Let $t_0\in(0,T)$, $\theta\in(t_0,T)$, and a non empty open interval $\omega\subset\subset\Omega$ be arbitrarily fixed. We restrict the function h in (1) to have support in $(t_0,T)\times\omega$ and the diffusion coefficients $c_j(\cdot)$, $j=1,2$, to be time–independent and to belong to the set $\mathcal{E}$ of positive piecewise smooth functions in Ω. The problem is to determine the discontinuous diffusion coefficients c_1, c_2 by observation data $U_2(\mathbf{u_0},h;\mathbf{c})|_{(t_0,T)\times\omega}$, $\mathbf{U}(\mathbf{u_0},h;\mathbf{c})|_{t=\theta}$. Then, from the

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knowledge of $\mathbf{c}$, we obtain the finite set of singularities of c_j: the interface $S^{c_j} \subset \Omega$, $j = 1, 2$. Around known $\widetilde{\mathbf{c}} = (\widetilde{c}_1, \widetilde{c}_2)$, we will determine $\mathbf{c} = (c_1, c_2)$, which means that we can know the solution $\mathbf{U}(\mathbf{u_0}, h; \mathbf{c})$ and S^{c_1}, S^{c_2}, which are *unknowns* of the inverse problem. In the formulation of the inverse problem, the initial values are also unknown. In the numerical part, a primal-dual path following interior-point method is used to recover $\mathbf{c}$. The method used in this work is similar as was used in [11]. For general references to interior-point methods see [18, 29, 6].

Throughout this paper we use the following notations:
for $s \in \mathbb{R}$, $L^2(H^s) = L^2(0, T; H^s(\Omega))$, $H^1(H^s) = H^1(0, T; H^s(\Omega))$.
The formal heat operator associated with a conductivity $\xi \in \mathcal{E}$ is written

$$\mathcal{L}(\xi)q = \partial_t q + \mathcal{A}(\xi)q,$$

where $\mathcal{A}(\xi) = -\partial_x(\xi \partial_x \cdot)$ is the formal spatial operator.
We define a self adjoint operator on $L^2(\Omega; \mathrm{d}x)$ by $A(\xi)p = \mathcal{A}(\xi)p$, with domain $D(A(\xi)) = \{p \in H_0^1(\Omega),\ \mathcal{A}(\xi)p \in L^2(\Omega)\}$. The solution $\mathbf{U}(\mathbf{u_0}, h; \mathbf{c})$ belongs to $C^0([0, T]; (L^2(\Omega))^2) \cap C^2((0, T]; D(A(c_1)) \times D(A(c_2)))$ for sufficiently smooth h. We need to introduce, for a positive number R, the sets:

$$\begin{aligned} \mathcal{E}(R) &= \{\xi \in \mathcal{E}; \|\xi_{|\Omega \setminus S^\xi}\|_{W^{1,\infty}(\Omega \setminus S^\xi)} \le R\}, \\ L_R^2 &= \{f \in L^2(\Omega);\ \|f\|_{L^2(\Omega)} \le R\}, \end{aligned}$$

where we set

$$\mathcal{E} = \{\xi : \overline{\Omega} \to (0, +\infty); \quad \xi \text{ is piecewise smooth}\}.$$

Finally, we assume that the coefficients a_{21} and a_{12} check the condition:

$$\textbf{(C-1)}: \quad \exists \delta > 0, \quad |a_{21}| \ge \delta > 0 \text{ and } |a_{12}| \ge \delta > 0 \text{ in } \omega.$$

Let us present the main theoretical result of this paper.

Theorem 1.1. (**Stability**) - *Let $R > 0$, $\widetilde{\mathbf{c}} \in \mathcal{E}^2$. Under* **(C-1)** *there exist $C > 0$, $h \in \mathcal{C}_c^\infty((t_0, T) \times \omega)$ such that, for all $\mathbf{c} \in \mathcal{E}(R)^2$, $\mathbf{u_0}, \tilde{\mathbf{u}}_\mathbf{0} \in (L_R^2)^2$, we have*

$$\begin{aligned} \|\mathbf{c} - \widetilde{\mathbf{c}}\|_{L^2(\Omega)} \quad \le \quad & C\Big(\sum_{k=1,2} \|\partial_t^k(u_2 - \widetilde{u}_2)\|_{L^2((t_0,T)\times\omega)} + \|\partial_t(\mathbf{u} - \widetilde{\mathbf{u}})|_{t=\theta}\|_{L^2(\omega)} \\ & + \|(\mathbf{u} - \widetilde{\mathbf{u}})|_{t=\theta}\|_{H^2(\omega)} + \|(\mathbf{u} - \widetilde{\mathbf{u}})|_{t=\theta}\|_{H^1(\Omega)} \Big). \end{aligned}$$

Corollary 1. (**Local Uniqueness**) - *Let $R > 0$, $\widetilde{\mathbf{c}} \in \mathcal{E}^2$. Under* **(C-1)** *there exists $h \in \mathcal{C}_c^\infty((t_0, T) \times \omega)$ such that, for all $\mathbf{c} \in \mathcal{E}(R)^2$, $\mathbf{u_0}, \tilde{\mathbf{u}}_\mathbf{0} \in (L_R^2)^2$, if $u_2 = \widetilde{u}_2$ in Q and $\mathbf{u}(\theta, \cdot) = \widetilde{\mathbf{u}}(\theta, \cdot)$ in Ω, then $\mathbf{c} = \widetilde{\mathbf{c}}$ and $\mathbf{u}_0 = \widetilde{\mathbf{u}}_0$.*

The domain of inverse problems and control problems for systems of coupled reaction-diffusion equations has recently been the subject of several papers (see e.g. [7, 2, 3, 8, 21, 22]). In all these papers, the diffusion coefficients are assumed to be smooth enough. Few results concern discontinuous diffusion coefficients. For the scalar case, we can cite [5, 19] where the authors established the uniqueness and stability results for both the discontinuous diffusion coefficients and the initial condition, from the measurement of the solution on an arbitrary part of the boundary, and for some arbitrary non negative time θ on all the domain Ω. For coupled reaction-diffusion coefficients with discontinuous conductivities, at our knowledge, any result exists.

The first part of this work is devoted to the proof of Theorem 1.1. We follow the same steps than in [19]: we show Carleman estimates for systems of heat equations (see Theorems 2.3, 2.4, Corollaries 3, 4) and for a stationary first-order equation (13). The difficulty that arises here for the case of a system is to deal with the same Carleman weight – the spatial function $\varphi(x)$ – adequate for any parabolic equation of the system. Carleman weight functions for one heat equation in the one-dimensional space are built in [4]. But we need two additional properties on φ when we deal with the first-order linear equation (13) where $\xi(x)$ is the unknown. Firstly, since $\rho\partial_x\varphi$ is sufficiently smooth for some positive function ρ, we are able to prove a stability result for (13) by using only one measurement. Secondly, the explicit form of φ in ω allows us to prove this stability result without the knowledge of c_1 and c_2 in ω. Hence the formulation of Theorem 1.1 only involves one measurement at $t = \theta$ without the knowledge of c_1, c_2 in ω.

In the numerical part of this work, we restore the discontinuous diffusion coefficients by solving an optimization problem with parabolic constraints. To this we apply a primal-dual path following interior-point method. Primal-dual interior-point methods have been used to solve optimal control problems with PDE constraints in [1, 10, 23, 25, 24, 26, 28, 30]. In particular, in [16, 17, 20], primal-dual interior-point methods were applied to optimal control problems with parabolic constraints. Up to our knowledge, besides the optimal control problems stated above, the similar studies of recovering the diffusion coefficients by applying primal-dual interior-point methods have not been proposed in the literature before.

The outline of the paper is as follows. In Section 2 we prove a Carleman estimate for the system with the observation of only one component, and a Carleman estimate for a stationary first order equation. We use these estimates to prove our main stability result for the diffusion coefficients $c_1(x)$ and $c_2(x)$. In Section 3 the primal-dual path following interior-point method is presented. Section 4 presents and discusses the results of the numerical experiments and Appendix contains the remaining proofs.

2. **Stability results.**

We introduce some notations. For $\xi \in \mathcal{E}$ let

$$
\begin{array}{rcl}
D(L(\xi)) & = & \{q \,:\, q \in L^2(0,T;H_0^1(\Omega)),\ \mathcal{L}(\xi)q \in L^2(Q)\}, \\
D(L^*(\xi)) & = & \{q \,:\, q \in L^2(0,T;H_0^1(\Omega)),\ \mathcal{L}^*(\xi)q \in L^2(Q)\}\ .
\end{array}
$$

The differential operators $\mathcal{L}(\xi)$, $\mathcal{L}^*(\xi)$ with respective domain $D(L(\xi))$, $D(L^*(\xi))$ are denoted by $L(\xi)$, $L^*(\xi)$. For any $r > 0$, the domain of the non-bounded self-adjoint operator $\mathbf{A}^r(\mathbf{c}) = (A^r(c_1), A^r(c_2))$ on $(L^2(\Omega))^2$ is $D(\mathbf{A}^r(\mathbf{c})) = D(A^r(c_1)) \times D(A^r(c_2))$.

We denote by S^ξ the total interface associated with $\xi \in \mathcal{E}$, that is,

$$S^\xi := \{x \in \overline{\Omega};\ \xi \text{ or } \partial_x\xi \text{ is discontinuous at } x\}.$$

Notice that S^ξ is a finite set. We write $S^{c_j} = S_j$, $\cup_{j=1}^2 S_j = S^{\mathbf{c}}$, $S^{\tilde{c}_j} = \tilde{S}_j$, $S^{\tilde{\mathbf{c}}} = \cup_{j=1}^2 \tilde{S}_j = \tilde{S}$.

For a piecewise smooth function q, we define $q^\pm(x) = \lim_{\varepsilon\searrow 0} q(x \pm \varepsilon)$, $[q]_x = q^+(x) - q^-(x)$.

If V is an open interval of $\mathbb{R}$ and f a function defined in $V' \supset V$, we write $f|_V \in C^k(\mathbb{R})$, $k \in [0,\infty]$, if $f|_V$ can be extended as a C^k function in $\mathbb{R}$.

Let us recall some tools used in [4] that provide a global Carleman estimate for *one* heat equation of the form

$$\left\{\begin{array}{rl} \partial_t q - \partial_x(\xi\partial_x q) + b\partial_x q + aq & = f \text{ in } (0,T)\times\Omega \\ q(t,\cdot) & = 0 \text{ on } (0,T)\times\partial\Omega, \\ q(0,\cdot) & \in L^2(\Omega). \end{array}\right. \tag{2}$$

We also consider the adjoint equation to (2):

$$\left\{\begin{array}{rl} -\partial_t q - \partial_x(\xi\partial_x q) - \partial_x(bq) + aq & = f \text{ in } (0,T)\times\Omega \\ q(t,\cdot) & = 0 \text{ on } (0,T)\times\partial\Omega, \\ q(T,\cdot) & \in L^2(\Omega). \end{array}\right. \tag{3}$$

The coefficients a, b are measurable scalar functions of t, x, with a, b, $\partial_x b$ bounded in Q, and $\xi \in \mathcal{E}$. Let us give the following simplified global Carleman estimate (see [4, Theorem 1.3]):

Proposition 1. *There exist a negative Lipschitzian function φ in $\overline{\Omega}$, and $C > 0$, $s_0 > 0$, such that, for all $s > s_0$ and for all q satisfying (2) or (3) with $f \in L^2(Q)$, we have*

$$\int_Q \Big(\Pi_{-1}|\partial_t q|^2 + \Pi_{-1}|A(\xi)q|^2 + \Pi_1|\partial_x q|^2 + \Pi_3|q|^2\Big)\,dx\,dt \tag{4}$$

$$\leq C\Big(\int_{(0,T)\times\omega} \Pi_3|q|^2\,dx\,dt + \int_Q \Pi_0|f|^2\,dx\,dt\Big),$$

where we set

$$\eta(t) = \frac{T^2}{t(T-t)}, \tag{5}$$

$$\Pi_r(t,x;s) = s^r\eta^r(t)e^{2s\eta(t)\varphi(x)},\ s\in\mathbb{R}^+,\ r\in\mathbb{R},\ (t,x)\in Q. \tag{6}$$

The "weight" function φ is not uniquely defined but satisfies the following properties.

1. φ is continuous and piecewise smooth. The set S^φ of discontinuities of $\partial_x\varphi$ is finite (this point is not crucial) and contains S^ξ,
2. $-C_2 \leq \varphi \leq -C_1 < 0$ in $\Omega = (0,1)$,
3. $\partial_x\varphi(x) \geq \delta > 0$ for $x \leq \inf\omega$, $x \notin S^\varphi$ and $\partial_x\varphi(x) \leq -\delta$ for $x \geq \sup\omega$, $x \notin S^\varphi$,
4. There exists a sufficiently large $K > 0$ such that $[\partial_x\varphi]_x \geq K$ at each $x \in S^\varphi$.

Remark 1. We assume that $\overline{\omega}\cap S^{\tilde{c}_j} = \emptyset$, $j = 1, 2$, since we always can reduce ω.

Remark 2. Let S' be any finite subset of $\Omega\setminus\overline{\omega}$ that contains S^ξ. Then one can choose φ such that $S^\varphi = S'$.

We call such functions φ, that satisfy the above conditions, "Carleman weight functions". It has to be notice that φ depends on the conductivity ξ.

With the same notations we obtain the following result.

Corollary 2. *Assume that in (2) (respectively, in (3)) the function f does not belong to $L^2(Q)$ but can be written*

$$f = f_0 + \partial_x f_1,$$

with $(f_0, f_1) \in (L^2(Q))^2$. Then there exist $C > 0$, $s_0 > 0$, such that for all $s > s_0$ and for all q satisfying (2) or (3) we have

$$\int_Q \Pi_3|q|^2\,dx\,dt \leq C\Big(\int_{(0,T)\times\omega} \Pi_3|q|^2\,dx\,dt + \int_Q (\Pi_0|f_0|^2 + \Pi_2|f_1|^2)\,dx\,dt\Big).$$

The proof is given in Appendix A.2

Remark 3. In Theorem 1, Lemma 2.1 and Corollary 2, we can replace the open set ω by any other nonempty open interval $\omega_0 = (\alpha_1, \alpha_2)$ such that $\omega_0 \subset\subset \omega$ and $\omega_0 \supset (\partial_x \varphi_{|\omega})^{-1}(\{0\})$.

We will need the following obvious but important lemma:

Lemma 2.1. *There exists $\rho \in \mathcal{E}$ such that*

$$\rho \partial_x \varphi \in W^{1,\infty}(\Omega).$$

Proof. We can choose ρ such that:

$$\rho = \frac{1}{|\partial_x \varphi|} \text{ in } \Omega \setminus (S^\varphi \cup \omega'),$$

where ω' is a open interval such that $\omega_0 \subset\subset \omega' \subset\subset \omega$. Then we extend smoothly ρ inside ω' so that $\rho \geq \delta > 0$. $\square$

For the sake of clarity we give a complete construction of φ in Ω. We assume that $\omega = (\alpha_1, \alpha_2)$ with $0 < \alpha_1 < \alpha_2 < 1$. Let S' be an arbitrary finite set of $\Omega \setminus \overline{\omega}$, that contains S^ξ. Let $\psi(x)$ be a Lipschitzian function in $\overline{\Omega}$ satisfying the following conditions.

1. $\psi\big|_{\partial\omega} = 0$, ψ is negative in $\Omega \setminus \overline{\omega}$, ψ is positive in ω.
2. ψ is increasing in $[0, \alpha_1[$, and decreasing in $[\alpha_2, 1]$.
3. $[\partial_x \psi]_y > 0$ at every point $y \in S'$.
4. $\psi|_V \in C^2(\mathbb{R})$, where V is any connected open subset of $\overline{\Omega} \setminus S'$.
5. $\psi^\pm(x) = 0$ implies $x \in \omega_0$.

Let $K > \max_{\overline{\Omega}} \psi$, $\lambda > 0$ and set

$$\varphi = e^{\lambda\psi} - e^{\lambda K}.$$

For any λ sufficiently large, φ is a Carleman weight function. See [13] for example.

Definition 2.2. We shall call such a function φ *adequate* for the equations (2) or (3).

We consider now a system of two heat equations

$$L(c_j)q_j = \sum_{k=1}^{2} (b_{jk}\partial_x q_k + a_{jk} q_k) + f_j,\ j = 1, 2, \tag{7}$$

or the adjoint system:

$$L^*(c_j)q_j = \sum_{k=1}^{2} (-\partial_x(q_k b_{kj}) + a_{kj} q_k) + f_j,\ j = 1, 2, \tag{8}$$

We assume that $\mathbf{c} = (c_1, c_2) \in \mathcal{E}^2$, a_{jk}, b_{jk}, $\partial_x b_{jk} \in L^\infty(Q)$, $f_j \in L^2(Q)$. We denote $S^{c_j} = S_j$, $j \in [[1, 2]]$, $S_1 \cup S_2 = S^{\mathbf{c}}$.
If a Carleman weight function is adequate for the two equations (7) or (8), we then call it a *common weight function* for the system (7) or (8). Thanks to Remark 2, there exists a common weight function φ, since we can choose $S' = S^{\mathbf{c}}$. The function φ is smooth in $\overline{\omega}$, and, thanks to Remark 3, we can assume that $\{x \in \overline{\Omega};\ \partial_x \varphi^\pm(x) = 0\} = \{x \in \omega;\ \partial_x \varphi(x) = 0\} \subset \omega_0$.
The main consequences lie in the following results.

Theorem 2.3. *There exist $C = C(Q, \omega, \mathbf{c}, \mathbf{a}, \mathbf{b}) > 0$, $s_0 = s_0(Q, \omega, \mathbf{c}, \mathbf{a}, \mathbf{b}) > 0$, such that for all $s > s_0$ and for all $\mathbf{q} = (q_1, q_2)$ satisfying (7) or (8) with $\mathbf{f} = (f_1, f_2) \in (L^2(Q))^2$, we have*

$$\int_Q \Big(\Pi_{-1}|\partial_t \mathbf{q}|^2 + \Pi_{-1}|\mathbf{A}(\mathbf{c})\mathbf{q}|^2 + \Pi_1|\partial_x \mathbf{q}|^2 + \Pi_3|\mathbf{q}|^2\Big)\,dx\,dt \le C\Big(\int_{(0,T)\times\omega_0} \Pi_3|\mathbf{q}|^2\,dx\,dt + \int_Q \Pi_0|\mathbf{f}|^2\,dx\,dt\Big), \tag{9}$$

where we set $\mathbf{a} = (a_{jk})_{jk}$, $\mathbf{b} = (\mathbf{b}_{jk})_{jk}$, and the weight functions Π_r are defined by (6).

Corollary 3. *Assume that $\mathbf{b}_{21} = 0$, $|a_{21}| \ge \delta > 0$ in ω for the system (7) (respectively, $\mathbf{b}_{12} = 0$, $|a_{12}| \ge \delta > 0$ in ω for the system (8)). Then there exist positive constants C, s_0 depending on $T, \Omega, \omega, \mathbf{c}, \mathbf{a}, \mathbf{b}, \delta$, such that, for all $s > s_0$, $\mathbf{f} \in (L^2(Q))^2$, $\mathbf{q}$ satisfying (7) (respectively, satisfying (8)), we have*

$$\int_Q \Big(\Pi_{-1}|\partial_t \mathbf{q}|^2 + \Pi_{-1}|\mathbf{A}(\mathbf{c})\mathbf{q}|^2 + \Pi_1|\partial_x \mathbf{q}|^2 + \Pi_3|\mathbf{q}|^2\Big)\,dx\,dt \le C\Big(\int_{(0,T)\times\omega_0} \Pi_7|q_k|^2\,dx\,dt + \int_Q \Pi_0|\mathbf{f}|^2\,dx\,dt\Big), \tag{10}$$

with $k = 2$ if $\mathbf{q}$ satisfies (7) (respectively, with $k = 1$ if $\mathbf{q}$ satisfies (8)).

The complete proof is in Appendix A.1. Let us consider the systems (7) or (8) with

$$f_j = f_{0j} + \partial_x f_{1j},\ j \in [[1, 2]],$$

where $f_{0j}, f_{1j} \in L^2(Q)$.

Theorem 2.4. *There exist positive constants C, s_0 such that, for all $s > s_0$, q_j satisfying (7) (respectively, (8)), we have*

$$\int_Q \Pi_3|\mathbf{q}|^2\,dx\,dt \le C\left(\int_Q (\Pi_2|\mathbf{f}_1|^2 + \Pi_0|\mathbf{f}_0|^2)\,dx\,dt + \int_{(0,T)\times\omega_0} \Pi_3|\mathbf{q}|^2\,dx\,dt\right), \tag{11}$$

where we set $\mathbf{q} = (q_1, q_2)$, $\mathbf{f}_1 = (f_{11}, f_{12})$, $\mathbf{f}_0 = (f_{01}, f_{02})$.

Proof: see Appendix, A.3. It combines the corresponding proof in [19] and [13, Proof of Theorem 5.1].

Corollary 4. *Assume that $\mathbf{b}_{21} = 0$, $|a_{21}| \ge \delta > 0$ in ω for the system (7) (respectively, $\mathbf{b}_{12} = 0$, $|a_{12}| \ge \delta > 0$ in ω for the system (8)). Then there exist positive constants C, s_0 depending on $T, \Omega, \omega, \mathbf{c}, \mathbf{a}, \mathbf{b}, \delta$, such that, for all $s > s_0$, $\mathbf{q}$ satisfying (7) (respectively, (8)), we have*

$$\int_Q \Pi_3|\mathbf{q}|^2\,dx\,dt \le C\left(\int_Q (\Pi_2|\mathbf{f}_1|^2 + \Pi_0|\mathbf{f}_0|^2)\,dx\,dt + \int_{(0,T)\times\omega_0} \Pi_7|q_k|^2\,dx\,dt\right), \tag{12}$$

where we set $k = 2$ for (7) (respectively, $k = 1$ for (8)).

The proof is in Appendix A.4

We consider now the following first-order partial differential equation with unknown function $\xi \in \mathcal{E}$:

$$-\partial_x(\xi \partial_x w) = f \text{ in } \Omega, \tag{13}$$

where $w \in H^1(\Omega)$, $f \in L^2(\Omega)$. Let $\tilde{\xi} \in \mathcal{E}$ be a known solution of (13) with (w, f) replaced by $(\tilde{w}, \tilde{f}) \in H^1(\Omega) \times L^2(\Omega)$. In fact we consider that $\tilde{\xi} = \tilde{c}_1$ or $\tilde{\xi} = \tilde{c}_2$, and, respectively, $\tilde{w}(x) = \tilde{u}_1(\theta, x)$ or $\tilde{w}(x) = \tilde{u}_2(\theta, x)$. We look for a $L^2(\Omega)$-estimate on $\gamma := \xi - \tilde{\xi}$, derived from a Carleman inequality. Let φ be a *common weight function* for the system (1), *with* $\tilde{\mathbf{c}} = (\tilde{c}_1, \tilde{c}_2)$ *replacing* $\mathbf{c}$. We set $S = S^{\xi}$, $\tilde{S} := S^{\tilde{\xi}}$, $\pi_r(x; s) = \Pi_r(\theta, x; s)$, where s is a large parameter, and Π_r is defined by (6). We recall that $\theta \in (0, T)$.

Our result is based on the method developed in [19]. We consider the following conditions on $\tilde{w}$:

(HS1): $\tilde{w} \in W^{1,\infty}(\Omega)$,

(HS2): $|\partial_x \tilde{w}| \geq \delta > 0$ a.e in $\Omega \setminus \omega_0$.

The following lemma will be proved in Appendix A.5

Lemma 2.5. *Let $R > 0$. Assume that $\xi \in \mathcal{E}(R)$, then, under* **(HS1)** *and* **(HS2)**, *there exist $C_1, s_0 > 0$ such that, $\forall s \geq s_0$, $\exists\ C_2(R; s) > 0$ so that*

$$\begin{aligned}\int_{\Omega \setminus \omega_0} \pi_3 |\gamma|^2 \, dx \quad &\leq \quad C_1 \Big(\int_{\Omega} \pi_1 |f - \tilde{f}|^2 \, dx + \int_{\omega_0} \pi_2 |\gamma|^2 \, dx \Big) \\ &\qquad + C_2(R, s) \|w - \tilde{w}\|^2_{H^1(\Omega)}.\end{aligned} \tag{14}$$

Remark 4. An assumption like $\tilde{w} = 0$ on Γ is superfluous.
The constants C_1, s_0, C_2 depend on $\tilde{\xi}$, φ, but not on ξ.
The constant C_1 does not depend on R, s.

The second following lemma completes Lemma 2.5 and gives a L^2 - estimate for γ in some neighbourhood of ω_0. We consider the following third condition on $\tilde{w}$:

(HS3): There exists an open interval ω_1 such that
1. $\omega_0 \subset\subset \omega_1 \subset\subset \omega$,
2. $\tilde{w}\big|_{\partial\omega_1}$ is constant,
3. $\tilde{w} \in C^2(\overline{\omega_1})$ and $|\partial_x \tilde{w}|^2 - (\tilde{w} - \tilde{w}\big|_{\partial\omega_1}) \partial_x^2 \tilde{w} \geq \delta > 0$ in ω_1.

Lemma 2.6. *Let $R > 0$, $\xi \in C^1(\overline{\omega_2})$ with $\|\xi\|_{W^{1,\infty}(\omega_1)} \leq R$. Then, under* **(HS3)**, *there exist C_3, $C_4(R) > 0$ such that*

$$\int_{\omega_1} |\gamma|^2 dx \quad \leq \quad C_3 \int_{\omega_1} |f - \tilde{f}|^2 \, dx + C_4(R) \|w - \tilde{w}\|^2_{H^2(\omega_1)}.$$

The constants C_3, C_4 do not depend on ξ.

Proof. Set $v = \tilde{w} - \tilde{w}|_{\partial\omega_1} \in H^1_0(\omega_1)$. Then $-\partial_x(\tilde{\xi}\partial_x v) = \tilde{f}$. Moreover $v \in H^2(\omega) \subset L^\infty(\omega)$. In ω_1, we have

$$-\partial_x(\gamma \partial_x v) = f - \tilde{f} + \partial_x(\xi \partial_x (w - \tilde{w})).$$

We multiply this equality by γv and we integrate by parts the first term in ω_1. We then obtain

$$\int_{\omega_1} \gamma^2 \Big(|\partial_x v|^2 - \frac{1}{4} \partial_x^2 (v^2) \Big) \, \mathrm{d}x = \int_{\omega_1} \Big(f - \tilde{f} + \partial_x(\xi \partial_x (w - \tilde{w})) \Big) \gamma v \, \mathrm{d}x.$$

Thanks to (HS3), we have

$$|\partial_x v|^2 - \frac{1}{4} \partial_x^2 (v^2) = \frac{1}{2} (|\partial_x v|^2 - v \partial_x^2 v) \geq \delta/2.$$

Hence

$$\int_{\omega_1} \gamma^2 \,\mathrm{d}x \le C(\delta) \int_{\omega_1} \Big(|f - \tilde{f}| + |\partial_x(\xi \partial_x (w - \tilde{w}))| \Big) |\gamma| \,\mathrm{d}x.$$

Then, thanks to Schwarz's inequality,

$$\int_{\omega_1} \gamma^2 \,\mathrm{d}x \le C \int_{\omega_1} \Big(|f - \tilde{f}| + |\partial_x(\xi \partial_x (w - \tilde{w}))| \Big)^2 \,\mathrm{d}x. \tag{15}$$

Since

$$\begin{aligned} \frac{1}{2} \|\partial_x(\xi \partial_x (w - \tilde{w}))\|_{L^2(\omega_1)} &\le \sup_{\omega_1} |\partial_x \xi| \, \|\partial_x (w - \tilde{w})\|_{L^2(\omega_1)} + \\ &\quad \sup_{\omega_1} |c| \|\partial_x^2 (w - \tilde{w})\|_{L^2(\omega_1)} \\ &\le R \|w - \tilde{w}\|_{H^2(\omega_1)}, \end{aligned}$$

the conclusion follows (15). □

Mixing lemmas 2.5 and 2.6, we straightforwardly obtain the following result.

Proposition 2. *Let $R > 0$. Assume that $\xi \in \mathcal{E}(R)$. Then, assuming* **(HS1) (HS2)**, **(HS3)**, *there exist C_1, $s_0 > 0$ such that, for all $s \ge s_0$, there exists $C_2(s) > 0$ satisfying*

$$\begin{aligned} \int_\Omega \pi_3 |\gamma|^2 \, dx &\le C_1 \Big(\int_{\Omega \setminus \omega} \pi_1 |f - \tilde{f}|^2 \, dx + \int_\omega \pi_2 |f - \tilde{f}|^2 \, dx \Big) \\ &\quad + C_2(R; s) \Big(\|w - \widetilde{w}\|^2_{H^1(\Omega)} + \|w - \widetilde{w}\|^2_{H^2(\omega)} \Big), \end{aligned}$$

where C_1, C_2 depend on δ, $\tilde{\xi}$ but not on (ξ, f, w).

Proof of the main result.

We use the following notations. For $V \subset \Omega$, we say that a function b defined in Ω belongs to $D(A(\xi))|_V$ for some $r > 0$ if it can be extended as a function in $D(A^r(\xi))$.

We set $S = S^{\mathbf{c}} = \cup_{j=1}^2 S_j$, $\mathbf{u} = (u_1, u_2) = \mathbf{U}(\mathbf{u}_0, h, \mathbf{c})$, $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2) = \mathbf{U}(\tilde{\mathbf{u}}_0, h, \tilde{\mathbf{c}})$. We consider a *common weight function* φ for the system (7) where $\tilde{\mathbf{c}}$ replaces $\mathbf{c}$.

Step 1. Firstly we show that we can build h such that, for all $\tilde{\mathbf{u}}_0 \in (L^2(\Omega)_R)^2$, $\tilde{u}_j$, $j = 1, 2$, satisfy the conditions (**HS2**) and (**HS3**).

Set $j \in [[1, 2]]$, $k \in \mathbb{N}$ and consider the conductivity c_j. We need the following result.

Lemma 2.7. *Let ω_1 be any non empty open interval such that $\omega_0 \subset\subset \omega_1 \subset \omega$. Then there exists a lipschitzian function $\tilde{b}_j \in D(A(c_j)) \cap D(A^{\frac{3}{2}}(c_j))\big|_\omega$ such that*

$$\tilde{b}_{j\ |\partial\omega_1} = \mathrm{Cste} > 0; \tag{16}$$

$$\partial_x \tilde{b}_j^{\pm}(x) \ge C > 0, \quad \forall x \le \alpha_1'; \tag{17}$$

$$\partial_x \tilde{b}_j^{\pm}(x) \le -C < 0, \quad \forall x \ge \alpha_2'; \tag{18}$$

$$|\partial_x \tilde{b}_j|^2 - (\tilde{b}_j - \tilde{b}_{j\ |\partial\omega}) \partial_x^2 \tilde{b}_j \ge C > 0 \ \ in\ \omega_1; \tag{19}$$

where $\omega_0 = (\alpha_1', \alpha_2')$.

Proof. Firstly, notice the following points. The domain $D(A(c_j))$ is the vectorial space of functions $f \in H_0^1(\Omega)$ such that $c_j \partial_x f \in H^1(\Omega)$. Since c_j is smooth in a neighborhoud of $\overline{\omega}$, if $f \in D(A(c_j))$, then $f\big|_\omega \in H^2(\omega)$, and f is continuously differentiable in ω. Moreover, for $r > 0$, $D(A^r(c_j))\big|_\omega = H^{2r}(\omega) \supset C^{2r}(\overline{\omega})$, $D(A^r(c_j))\big|_{\omega_1} = H^{2r}(\omega_1) \supset C^{2r}(\overline{\omega_1})$. Since the spatial dimension is one, if $f \in H^3(\omega)$, then f is twice continuously differentiable in ω, and if $f \in H^3(\omega_1) \cap H^3(\omega \setminus \overline{\omega_1})$, then $f \in H^3(\omega)$ if and only if $[f]_y = [\partial_x f]_y = [\partial_x^2 f]_y = 0$ for all $y \in \partial\omega_1$.

Set $\omega_1 = (\alpha_3, \alpha_4)$. Let $r > 0$, and g_1, g_2 be two smooth positive functions in $\mathbb{R}$ satisfying

$$\frac{g_1(\alpha_3)}{c_j(\alpha_3)} = \frac{g_2(\alpha_4)}{c_j(\alpha_4)} = r(\alpha_4 - \alpha_3), \tag{20}$$

$$\int_0^{\alpha_3} g_1(y)\frac{dy}{c_j(y)} = \int_{\alpha_4}^1 g_2(y)\frac{dy}{c_j(y)} = 1. \tag{21}$$

Put

$$b(x) = \begin{cases} \displaystyle\int_0^x g_1(z)\frac{dz}{c_j(z)} & , \quad x \in [0, \alpha_3], \\ r(x - \alpha_3)(\alpha_4 - x) + 1 & , \quad x \in [\alpha_3, \alpha_4], \\ \displaystyle\int_x^1 g_2(z)\frac{dz}{c_j(z)} & , \quad x \in [\alpha_4, 1]. \end{cases}$$

Then b satisfies conditions (16), (17), (18), (19), with $b, c_j\partial_x b \in C^2(\overline{\Omega \setminus \omega_1})$, $b|_{\omega_1} \in C^\infty(\mathbb{R})$, $[b]_y = [\partial_x b]_y = 0$, $\forall y \in \partial\omega_1$.
Hence b belongs to $D(A(c)) \cap D(A^{\frac{3}{2}}(c))|_{\omega_1} \cap D(A^{\frac{3}{2}}(c))|_{\Omega\setminus\overline{\omega_1}}$, but $b \notin D(A^{\frac{3}{2}}(c))|_\omega$ since the property $([\partial_x^2 b]_y = 0,\ \forall y \in \partial\omega_1)$ fails. We then consider $\tilde{b}_j$ as a regularization of b near ω_1, sufficiently closed to b in $W^{2,\infty}(\omega_1)$ and such that $\tilde{b}_j = b$ on $\partial\omega_1$. □

Set ω_1 be a non empty open interval such that $\omega_0 \subset\subset \omega_1 \subset\subset \omega$, $\tilde{b}_j$, $j = 1, 2$, as in Lemma 2.7, and $\tilde{\mathbf{b}} = (\tilde{b}_1, \tilde{b}_2)$. As in [19, Proof of Corollary 3.1], the set $G := \{\mathbf{U}(0, h, \mathbf{c})(\theta),\ h \in \mathcal{C}_0^\infty((t_0, T) \times \omega_0)\}$ is dense in $D(\mathbf{A}^r(\mathbf{c}))$ equipped with the norm $\sum_{j=1,2} \|A^r(c_j)p_j\|_{L^2(\Omega)}$, for all $r \in \mathbb{R}^+$. This comes thanks to (3) applied to (8), the system (1) is approximatively controllable in $D(\mathbf{A}^r(\mathbf{c}))$. Then, for any $\varepsilon > 0$, there exists $h \in \mathcal{C}_0^\infty((t_0, T) \times \omega_0)$ such that

$$\|\mathbf{U}(\mathbf{0}, \varepsilon h, \tilde{\mathbf{c}})|_{t=\theta} - \tilde{\mathbf{b}}\|_{D(A(c_j))} + \|\mathbf{U}(\mathbf{0}, \varepsilon h, \tilde{\mathbf{c}})|_{t=\theta} - \tilde{\mathbf{b}}\|_{D(A^{\frac{3}{2}}(c_j))|_\omega} < \varepsilon,\ j = 1, 2. \tag{22}$$

Observe that if $p \in D(A(c_j))$ then $p\big|_{\Omega\setminus S_j} \in H^2(\Omega \setminus S_j)$. See for example [5, Prop. A.4] for a proof. Moreover, $[p]_y = 0$ for all $y \in S_j$. Hence $p \in W^{1,\infty}(\Omega)$, since, in the one dimensional case, $H^1(\Omega) \subset C^0(\overline{\Omega}) \subset L^\infty(\Omega)$. Thus, $D(A(c_j)) \subset W^{1,\infty}(\Omega)$ with continuous embedding. Furthermore, we have $D(A^{\frac{3}{2}}(c_j))|_\omega = H^3(\omega) \subset W^{2,\infty}(\omega)$, with continuous embedding. From (22), we thus have

$$\|\mathbf{U}(\mathbf{0}, \varepsilon h, \tilde{\mathbf{c}})|_{t=\theta} - \tilde{\mathbf{b}}\|_{W^{1,\infty}(\Omega)} + \|\mathbf{U}(\mathbf{0}, \varepsilon h, \tilde{\mathbf{c}})|_{t=\theta} - \tilde{\mathbf{b}}\|_{W^{2,\infty}(\omega)} < \varepsilon,\ j = 1, 2. \tag{23}$$

Here, $\tilde{\mathbf{b}} = (\tilde{b}_1, \tilde{b}_2)$. Let $\tilde{\mathbf{u}}_0 \in L_R^2$ and put

$$\mathbf{p} = (p_1, p_2) = \varepsilon\mathbf{U}(\tilde{\mathbf{u}}_0, h, \tilde{\mathbf{c}})|_{t=\theta}.$$

Observing that $\mathbf{U}(\cdot, \cdot, \tilde{\mathbf{c}})$ is linear in the two first variables, we thus have

$$\mathbf{p} = \tilde{\mathbf{b}} + (\mathbf{U}(0, \varepsilon h, \tilde{\mathbf{c}})|_{t=\theta} - \tilde{\mathbf{b}}) + \varepsilon\mathbf{U}(\tilde{\mathbf{u}}_0, 0, \tilde{\mathbf{c}})|_{t=\theta}.$$

Since

$$\|\mathbf{U}(\tilde{\mathbf{u}}_0, 0, \tilde{\mathbf{c}})|_{t=\theta}\|_{W^{1,\infty}(\Omega)} + \|\mathbf{U}(\tilde{\mathbf{u}}_0, 0, \tilde{\mathbf{c}})|_{t=\theta}\|_{W^{2,\infty}(\omega)} < C(R),$$

where $C(R)$ does not depend on $\tilde{\mathbf{u}}_0$, then, thanks to (16), (17), (18), (19), (23), for all $\varepsilon > 0$ sufficiently small, for $j = 1, 2$, there exist an open interval ω_1^j and a positive constant C such that $\omega_0 \subset\subset \omega_1^j \subset\subset \omega$, and

$$p_{j\ |\partial\omega_1^j} = \text{Cste} > 0, \tag{24}$$

$$\partial_x p_j \geq C > 0 \ \text{ a.e. } \ x < \alpha_1', \tag{25}$$

$$\partial_x p_j \leq -C < 0 \ \text{ a.e. } \ x > \alpha_2', \tag{26}$$

$$|\partial_x p_j|^2 - (p_j - p_j|_{\partial\omega^j})\partial_x^2 p_j \geq C > 0 \ \text{ in } \omega_1^j. \tag{27}$$

Hence, $\varepsilon^{-1}\mathbf{p} = \mathbf{U}(\tilde{\mathbf{u}}_\mathbf{0}, h, \tilde{\mathbf{c}})|_{t=\theta}$ satisfies (24), (25), (26), (27) (with other constants), and so, for all $j = 1, 2$, $\tilde{u}_j$ satisfies (**HS2**) and (**HS3**).

Step 2. For all $\varepsilon > 0$, the function $\tilde{\mathbf{u}}$ belongs to $H^3(\varepsilon, T; D(\mathbf{A}^r(\mathbf{c})))$ with arbitrary large r. Thus $\partial_t^k \tilde{\mathbf{u}} \in (L^\infty(\varepsilon, T; W^{1,\infty}(\Omega)))^2$, $k = 0, 1, 2$, and

$$\|\partial_t^k \tilde{u}_j\|_{L^\infty(\varepsilon,T;W^{1,\infty}(\Omega))} \leq C(R), \ k = 0, 1, 2, \ j = 1, 2,$$

since the choice of h depends on R but not on $\tilde{\mathbf{u}}_0$. For the same reason and since $\mathbf{c} \in \mathcal{E}(R)^2$, we also have

$$\|\partial_t^k u_j\|_{L^\infty(\varepsilon,T;W^{1,\infty}(\Omega))} \leq C(R), \ k = 0, 1, 2, \ j = 1, 2.$$

Since we can consider the system (1) starting at time t_0 we can replace t_0 and ε by 0 and omit it in the proof.
We set $v_j(x) := u_j(\theta, x)$, $\tilde{v}_j(x) := \tilde{u}_j(\theta, x)$, $z_j := \partial_t u_j - \partial_t \tilde{u}_j$, $w_j = \partial_t z_j$, $\gamma_j := c_j - \tilde{c}_j$. Notice that the following conditions are satisfied:

(**H1**): $\forall j \in [[1, 2]]$, $\tilde{v}_j$ satisfies (**HS1**).
(**H2**): $\forall j \in [[1, 2]]$, $\tilde{v}_j$ satisfies (**HS2**).
(**H3**): $\forall j \in [[1, 2]]$, $\tilde{v}_j$ satisfies (**HS3**).
(**H4**): $\partial_t^k u_j$, $\partial_t^k \tilde{u}_j \in L^2(W^{1,\infty})$, $j = 1, 2$, $k = 0, 1, 2$.

We have:

$$\left\{ \begin{array}{rcl} \partial_t z_j - \partial_x(\tilde{c}_j \partial_x z_j) & = & \sum_{k=1}^2 a_{jk} z_k + \partial_x(\gamma_j \partial_x \partial_t u_j) \text{ in } Q, \\ z_j(t, x) & = & 0 \text{ on } \Sigma, \end{array} \right.$$

By differentiating (28) according to t we have:

$$\left\{ \begin{array}{rcl} \partial_t w_j - \partial_x(\tilde{c}_j \partial_x w_j) & = & \sum_{k=1}^2 a_{jk} w_k + \partial_x(\gamma_j \partial_x \partial_t^2 u_j) \text{ in } Q, \\ w_j(t, x) & = & 0 \text{ on } \Sigma. \end{array} \right.$$

We also have the following equalities:

$$\partial_x(c_j \partial_x v_j) \quad = \quad f_j \equiv \partial_t u_j(\theta) - \sum_k a_{jk} v_k - \delta_{2j} h(\theta), \tag{28}$$

$$\partial_x(\tilde{c}_j \partial_x \tilde{v}_j) \quad = \quad \tilde{f}_j \equiv \partial_t \tilde{u}_j(\theta) - \sum_k a_{jk} \tilde{v}_k - \delta_{2j} h(\theta). \tag{29}$$

We set

$$
\begin{aligned}
B_{j1} &:= \int_\Omega \frac{1}{2}\pi_1 |z_j(\theta)|^2 \,\mathrm{d}x,\\
B_1 &:= \sum_{j=1,2} B_{j1}.
\end{aligned}
$$

Lemma 2.8. *Under (**H1**)–(**H4**), there C_1, $s_0>0$ such that, for all $s>s_0$, there exists $C_2(s)>0$ satisfying:*

$$
B_1 \;\le\; C_1\int_\Omega \pi_1|\overline{\gamma}|^2\,dx + C_2(s)\Big(\int_{(0,T)\times\omega}(|z_2|^2+|w_2|^2)\,dx\,dt\Big). \tag{30}
$$

$$
\begin{aligned}
B_1 \;\ge\; & C^{-1}\int_\Omega \pi_3|\overline{\gamma}|^2\,dx - C_2(s)\Big(\int_\omega |\mathbf{z}(\theta)|^2\,dx + \|\mathbf{v}-\tilde{\mathbf{v}}\|^2_{H^1(\Omega)} \\
& +\|\mathbf{v}-\tilde{\mathbf{v}}\|^2_{H^2(\omega)}\Big),
\end{aligned} \tag{31}
$$

where $\overline{\gamma}=(\gamma_1,\gamma_2)$. Recall that we set $\pi_r(x):=\Pi_r(\theta,x;s)$ and the constants may depend on R.

See the proof in Appendix A.6.

Thanks to (30), (31) we obtain

$$
\begin{aligned}
\int_\Omega \pi_3|\gamma|^2\,\mathrm{d}x \;\le\; & C(s)\Big(\int_{(0,T)\times\omega}(|z_2|^2+|w_2|^2)\,\mathrm{d}x\,\mathrm{d}t + \int_\omega |\mathbf{z}(\theta)|^2\,\mathrm{d}x \\
& +\|\mathbf{v}-\tilde{\mathbf{v}}\|^2_{H^2(\omega)} + \|\mathbf{v}-\tilde{\mathbf{v}}\|^2_{H^1(\Omega)}\Big),
\end{aligned}
$$

for all $s>s_0$. The conclusion follows.

3. **Numerical algorithm.** We begin the derivation of the primal-dual path following interior-point method by presenting the constrained minimization problem related to recovering $\mathbf{c}$.

Based on the theory presented in Section 2 and in particular on Theorem 1.1 and Corollary 1 reconstructing two discontinuous diffusion coefficients amounts to the minimization of the following functional

$$
\min\left\{\frac{1}{2}\|\partial_t(u_2-\tilde{u}_2)\|_{L^2(\omega_{t_0,T})} + \frac{1}{2}\|\partial_t^2(u_2-\tilde{u}_2)\|_{L^2(\omega_{t_0,T})}\right\} \tag{32}
$$

subject to

$$
\begin{cases}
\partial_t u_1 - \partial_x(c_1\partial_x)u_1 &= a_{11}u_1 + a_{12}u_2 \text{ in Q},\\
\partial_t u_2 - \partial_x(c_2\partial_x)u_2 &= a_{21}u_1 + a_{22}u_2 \text{ in Q},\\
u_j(t,x) &= 0 \text{ on } \Sigma,\ j=1,2,\\
u_j(0,x) &= u_{0,j}(x) \text{ in } \Omega.
\end{cases}
$$

and $\mathbf{u}(\theta,\cdot)=\widetilde{\mathbf{u}}(\theta,\cdot)$ in Ω .

The minimization of (32) can be reformulated into a QP form as follows:

$$
\min\left\{\frac{1}{2}u_2^T(D^TD+D_2^TD_2)u_2 - u_2^T(D^TD+D_2^TD_2)\tilde{u}_2 + \frac{1}{2}\tilde{u}_2^T(D^TD+D_2^TD_2)\tilde{u}_2\right\}
$$

where we have denoted $\partial_t =: D$ and $\partial_t^2 =: D_2$. Note that the functional in (32) depends only on u_2 and the aim is to recover $\mathbf{c}=(c_1(x),c_2(x))$ hence we have both the $\mathbf{u}=(u_1(t,x),u_2(t,x))$ and $\mathbf{c}=(c_1(x),c_2(x))$ as the unknowns in our QP problem. This implies that (1) becomes non-linear with respect to solving the QP optimization problem. Hence in recovering the discontinuous diffusion coefficients we consider the linearized version of (1)

$$\left\{\begin{array}{rcl} \partial_t u_1 - \partial_x(\tilde{c}_1\partial_x)u_1 - a_{11}u_1 - a_{12}u_2 - \partial_x(c_1\partial_x)\tilde{u}_1 &=& -\partial_x(\tilde{c}_1\partial_x)\tilde{u}_1 \quad \text{in Q}, \\ \partial_t u_2 - \partial_x(\tilde{c}_2\partial_x)u_2 - a_{21}u_1 - a_{22}u_2 - \partial_x(c_2\partial_x)\tilde{u}_2 &=& -\partial_x(\tilde{c}_2\partial_x)\tilde{u}_2 \quad \text{in Q}, \\ u_j(t,x) &=& 0 \quad \text{on } \Sigma,\ j=1,2, \\ u_j(0,x) &=& u_{0,j}(x) \quad \text{in } \Omega. \end{array}\right. \tag{33}$$

Next we denote (33) together with the constraint $\mathbf{u}(\theta,\cdot) = \tilde{\mathbf{u}}(\theta,\cdot)$ by $\mathbf{R}\mathbf{x} = r$ and write the constraint minimization problem as follows

$$\min\left(\frac{1}{2}\mathbf{x}^T K\mathbf{x} + k^T\mathbf{x} + \kappa\right) \qquad \text{s.t. } \mathbf{x} \in \mathcal{F}_1 \tag{34}$$

where $\mathbf{x} = \begin{bmatrix} u_1 \\ u_2 \\ c_1 \\ c_2 \end{bmatrix}$, $K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & D^TD + D_2^TD_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $k = \begin{bmatrix} 0 \\ -(D^TD + D_2^TD_2)\tilde{u_2} \\ 0 \\ 0 \end{bmatrix}$,

$\kappa = \frac{1}{2}\tilde{u}_2^T(D^TD + D_2^TD_2)\tilde{u}_2$, $\quad \mathcal{F}_1 = \{\mathbf{x} \mid \mathbf{R}\mathbf{x} = r,\ \ell \le \mathbf{x} \le \nu\}$, $\ell = \begin{bmatrix} -M \\ -M \\ c_1^{\min} \\ c_2^{\min} \end{bmatrix}$ and

$\nu = \begin{bmatrix} M \\ M \\ c_1^{\max} \\ c_2^{\max} \end{bmatrix}$, $(c_1^{\min}, c_2^{\min}) > 0$, $(c_1^{\max}, c_2^{\max}) < R$ and $M >> 0$ so that the constraint on $\mathbf{u} = (u_1, u_2)$ is practically ineffective.

We start the derivation of the interior point method by introducing slack variables in order to replace the inequality constraints with simpler non-negativity constraints (the non-negativity constraints are enforced separately) thus the primal problem becomes

$$\min\left(\frac{1}{2}\mathbf{x}^T K\mathbf{x} + k^T\mathbf{x} + \kappa\right) \qquad \text{s.t. } \mathbf{x} \in \mathcal{F} \tag{35}$$

$\mathcal{F} = \{\mathbf{x} |\ \mathbf{R}\mathbf{x} = r,\ \mathbf{x} - \ell = g,\ \nu - \mathbf{x} = p\}$ and $g, p \ge 0$, where $\mathbf{x}, g$ and p are vectors in $\mathbb{R}^{n_1}$, r is a vector in $\mathbb{R}^{n_2}$, K is an $n_1 \times n_1$ matrix and $\mathbf{R}$ is an $n_2 \times n_1$ matrix. The dual of (35) is

$$\begin{array}{ll} \max & \left(\kappa + r^Ty + \ell^Tz - \nu^Tn - \dfrac{1}{2}\mathbf{x}^TK\mathbf{x}\right) \\ \text{s.t.} & \mathbf{R}^Ty + z - n - K\mathbf{x} = r \\ & z, n \ge 0,\ y \quad \text{free}, \end{array} \tag{36}$$

where y is a vector in $\mathbb{R}^{n_2}$ and z and n are vectors in $\mathbb{R}^{n_1}$. The dual variables z, n are complementary to the nonnegative primal variables g, p which implies that $z, n \ge 0$.

Next we define the central path which yields the path that is followed in our approach. The central path, parameterized by μ, can be defined as
$\mathcal{P}\{(x_\mu, g_\mu, p_\mu, y_\mu, z_\mu, n_\mu) \mid \mu > 0\}$. Each $\mu > 0$ define the associated central path

point in the primal-dual space that satisfies simultaneously the conditions of primal feasibility, dual feasibility and μ-complementarity, namely the conditions

$$\begin{aligned} \mathbf{R}^T y + z - n - K\mathbf{x} &= k \\ \mathbf{R}\mathbf{x} &= r \\ \mathbf{x} - g &= \ell \\ \mathbf{x} + p &= \nu \\ GZ\mathbf{1} &= \mu\mathbf{1} \\ PN\mathbf{1} &= \mu\mathbf{1} \end{aligned} \tag{37}$$

where $\mathbf{1}$ is a vector of all ones, G, Z, P and N are diagonal matrices with elements g_i, z_i, p_i, n_i respectively. These conditions are the optimality conditions, often known as *Karush-Kuhn-Tucker* (KKT) conditions, for a Fiacco-McCormick [9] type logarithmic barrier formulation of problem (34). The logarithmic barrier formulation of (34) writes

$$\begin{aligned} \min \quad & \frac{1}{2}\mathbf{x}^T K\mathbf{x} + k^T\mathbf{x} + \kappa - \mu\left(\sum_i \log g_i + \sum_i \log p_i\right) \\ \text{s.t} \quad & \mathbf{R}\mathbf{x} = r. \end{aligned} \tag{38}$$

As $\mu \to 0$ the trajectory $\mathcal{P}$ converges to the optimal solution of both the primal and dual problem. At the optimal point $(x^*, g^*, p^*, y^*, z^*, n^*)$, $\mu \equiv 0$ and the primal objective function is equivalent with the dual objective function. Further this means that (38) is equivalent with (34) and the conditions (37) are the KKT conditions for the original problem (34).

When K is positive semidefinite these KKT conditions are both necessary and sufficient optimality conditions for the QP problem [18] hence we can solve the QP problem (34) by finding a solution to the system (37). We write the KKT conditions in a form on a mapping F from $\mathbb{R}^{5n_1 \times n_2}$ to $\mathbb{R}^{5n_1 \times n_2}$ by

$$F(\mathbf{x}, g, p, y, z, n; \mu) = \begin{bmatrix} K\mathbf{x} - \mathbf{R}^T y - z + n + k \\ r - \mathbf{R}\mathbf{x} \\ \ell - \mathbf{x} + g \\ \mathbf{x} + p - \nu \\ GZ\mathbf{1} - \mu\mathbf{1} \\ PN\mathbf{1} - \mu\mathbf{1} \end{bmatrix} = 0. \tag{39}$$

Assuming that μ is fixed and applying Newton's method to (39) we obtain

$$\begin{bmatrix} -K & 0 & 0 & \mathbf{R}^T & I & -I \\ \mathbf{R} & 0 & 0 & 0 & 0 & 0 \\ I & -I & 0 & 0 & 0 & 0 \\ I & 0 & I & 0 & 0 & 0 \\ 0 & G^{-1}Z & 0 & 0 & I & 0 \\ 0 & 0 & P^{-1}N & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x} \\ \Delta g \\ \Delta p \\ \Delta y \\ \Delta z \\ \Delta n \end{bmatrix} = \begin{bmatrix} \Phi \\ \Lambda \\ \Lambda_g \\ \Lambda_p \\ \Gamma_z \\ \Gamma_n \end{bmatrix}$$

where

$$\begin{aligned}
\Phi &= k + K\mathbf{x} - \mathbf{R}^T y - z + n \\
\Lambda &= r - \mathbf{R}\mathbf{x} \\
\Lambda_g &= \ell - \mathbf{x} - g \\
\Lambda_p &= \nu - \mathbf{x} - p \\
\Gamma_z &= \mu G^{-1}\mathbf{1} - z - G^{-1}\Delta g^T \Delta z \\
\Gamma_n &= \mu P^{-1}\mathbf{1} - n - P^{-1}\Delta p^T \Delta n
\end{aligned}$$

We can eliminate $\Delta p, \Delta g, \Delta z$ and Δn without producing any off-diagonal fill-ins in the remaining system hence resulting into a reduced KKT system given by

$$\begin{bmatrix} -(K + NP^{-1} + G^{-1}Z) & \mathbf{R}^T \\ R & 0 \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x} \\ \Delta y \end{bmatrix} = \begin{bmatrix} \widetilde{\Phi} \\ \Lambda \end{bmatrix},$$

where $\widetilde{\Phi} = \Phi - \Gamma_n - \Gamma_z - NP^{-1}\Lambda_p - G^{-1}Z\Lambda_g$ and

$$\begin{aligned}
\Delta z &= \Gamma_z + G^{-1}Z\left(\Lambda_g - \Delta\mathbf{x}\right) \\
\Delta n &= \Gamma_n + NP^{-1}\left(\Delta\mathbf{x} - \Lambda_p\right) \\
\Delta g &= GZ^{-1}\left(\Gamma_z - \Delta z\right) \\
\Delta p &= N^{-1}P\left(\Gamma_n - \Delta n\right)
\end{aligned}$$

The algorithm for solving the problem is based on Mehrotra's [15] predictor-collector method and it proceeds iteratively from an initial point $(\mathbf{x}^0, g^0 . p^0, y^0, z^0, n^0)$ through a sequence of points determined from the search directions described above

$$\begin{aligned}
\mathbf{x}^{k+1} &= \mathbf{x}^k + \alpha_{\text{primal}}\Delta\mathbf{x} \\
g^{k+1} &= g^k + \alpha_{\text{primal}}\Delta g \\
p^{k+1} &= p^k + \alpha_{\text{primal}}\Delta p \\
y^{k+1} &= y^k + \alpha_{\text{dual}}\Delta y \\
z^{k+1} &= z^k + \alpha_{\text{dual}}\Delta z \\
n^{k+1} &= n^k + \alpha_{\text{dual}}\Delta n
\end{aligned}$$

The step lengths α_{primal} and α_{dual} are obtained as follows; first we compute the maximal feasible step length in order to enforce the non-negativity requirements $(g^{k+1}, p^{k+1}, z^{k+1}, n^{k+1}) > 0$

$$\begin{aligned}
\alpha_{\text{p}}^{\max} &= 0.95 \times \min\left(\min\left(-\frac{g}{\Delta g}\right), \min\left(-\frac{p}{\Delta p}\right), 1\right) \\
\alpha_{\text{d}}^{\max} &= 0.95 \times \min\left(\min\left(-\frac{z}{\Delta z}\right), \min\left(-\frac{n}{\Delta n}\right), 1\right).
\end{aligned}$$

Then we set $\alpha_{\text{dual}} = \alpha_{\text{p}}^{\max}$ and compute α_{primal} by using a backtracking line search on the interval $\left[0, \alpha_{\text{p}}^{\max}\right]$. As the choice of the duality measure μ we use a similar formulation as proposed in [27].

4. **Numerical examples.** The observation data $U_2(\mathbf{u_0}, h; \mathbf{c})|_{\omega_{t_0,T}}$, $\mathbf{U}(\mathbf{u_0}, h; \mathbf{c})|_{t=\theta}$ was simulated as follows. T was set to $T = 1$ and Gaussian noise $\epsilon_{\text{noise}} \sim \mathcal{N}(0, \sigma^2 I)$ with a standard deviation of 1% of the maximum of the noiseless observations was added to the simulated data. A more dense grid was used for the generation of the data than was used in any of the computations of the diffusion coefficients, thus avoiding the inverse grime.

The problem of recovering the discontinuous diffusion coefficients related to the optimization problem

$$\min \quad \left\{ \frac{1}{2}\|D(u_2 - \tilde{u}_2)\|^2_{L^2(\omega_{t_0,T})} + \frac{1}{2}\|D_2(u_2 - \tilde{u}_2)\|^2_{L^2(\omega_{t_0,T})} \right\}$$

subject to

$$\begin{cases} \partial_t u_1 - \partial_x(\tilde{c}_1 \partial_x)u_1 - a_{11}u_1 - a_{12}u_2 - \partial_x(c_1\partial_x)\tilde{u}_1 & = & -\partial_x(\tilde{c}_1\partial_x)\tilde{u}_1 \quad \text{in Q}, \\ \partial_t u_2 - \partial_x(\tilde{c}_2 \partial_x)u_2 - a_{21}u_1 - a_{22}u_2 - \partial_x(c_2\partial_x)\tilde{u}_2 & = & -\partial_x(\tilde{c}_2\partial_x)\tilde{u}_2 \quad \text{in Q}, \\ u_j(t,x) & = & 0 \quad \text{on } \Sigma,\ j = 1,2, \\ u_j(0,x) & = & u_{0,j}(x) \quad \text{in } \Omega. \\ \tilde{\mathbf{u}}(\theta, x) & = & \mathbf{u}(\theta, x) \end{cases}$$

was discretized as follows. Let $t \in [0, T], x \in [0, 1]$ and let N_t denote the number of steps in time and N_x the number of steps in space. N_t and N_x are related to n_1 and n_2 by $n_1 = 2(N_x \times N_t + N_x + 1)$ and $n_2 = 4(N_x \times N_t)$, respectively. We approximate the time derivative ∂_t by the explicit Euler method and the spatial derivatives ∂_x by the finite difference method. In the following calculations we use $N_x = 128$ and $N_t = 80$ and consider two different piecewise regular realizations of the $\tilde{\mathbf{c}}$.

In order to study the accuracy of the computed reconstructions quantitatively we compute the relative errors of $c_1(x)$ and c_2(x) in Ω with respect to $\widetilde{c}_1(x)$ and $\widetilde{c}_2(x)$, respectively

$$\delta_{\mathrm{c_j}} = \frac{\|\widetilde{c}_j(x) - c_j(x)\|_{L^2}}{\|\widetilde{c}_j(x)\|_{L^2}}, \qquad j = 1, 2.$$

Figures 1 and 2 show the reconstructed piecewise regular diffusion coefficient for two different test cases. The relative errors related to the reconstructions in figure 1 are $\delta_{c_1} = 0.1541$ and $\delta_{c_2} = 0.1127$. The corresponding relative errors related to figure 2 are $\delta_{c_1} = 0.1443$ and $\delta_{c_2} = 0.0802$.

The numerical simulations indicate that the primal-dual path following interior-point method with the observation data $U_2(\mathbf{u_0}, h; \mathbf{c})|_{\omega_{t_0,T}}$, $\mathbf{U}(\mathbf{u_0}, h; \mathbf{c})|_{t=\theta}$ of Theorem 1.1 allow an accurate reconstruction of the discontinuous diffusion coefficients c_1 and c_2.

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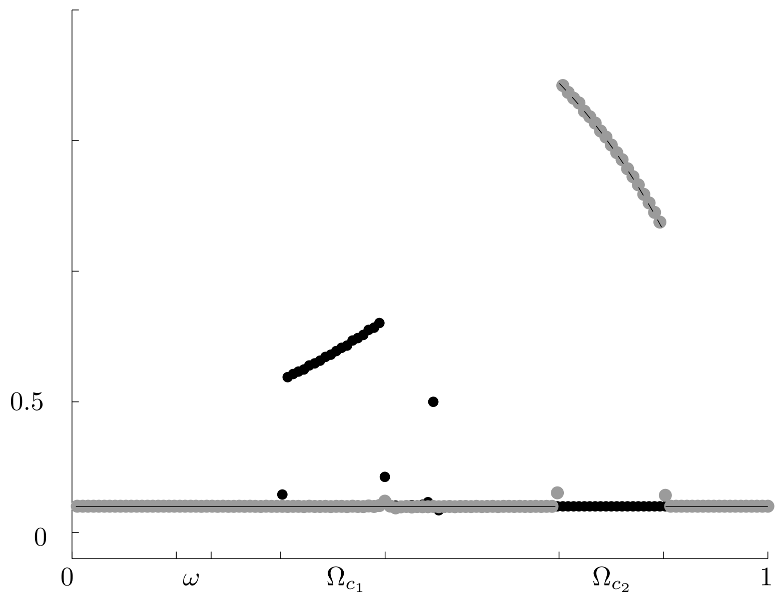

FIGURE 1. Reconstructions of c_1 and c_2 . The results for c_1 are denoted by black dots and the results for c_2 are denoted by gray dots. $\widetilde{c}_1$ is denoted by solid line and $\widetilde{c}_2$ by a dashed line. The relative error δ_{error} for $c_1 = 0.1541$ in Ω and for $c_2 = 0.1127$ in Ω.

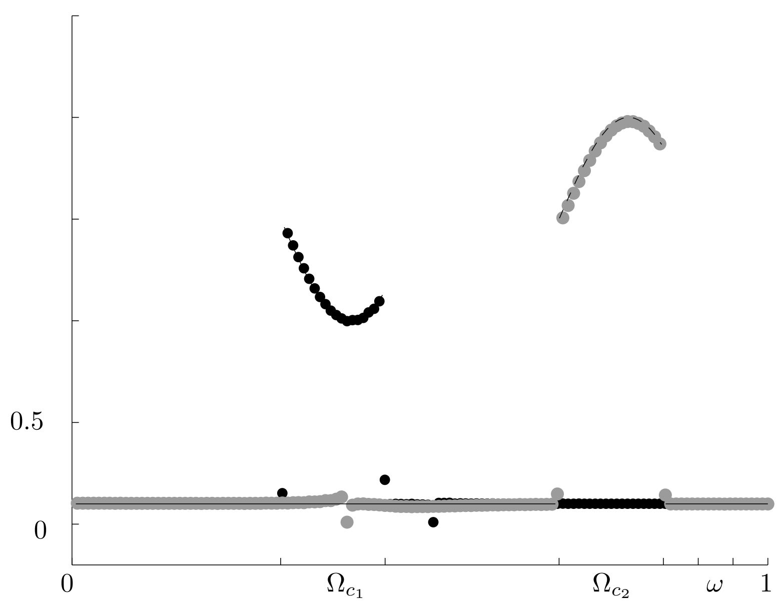

FIGURE 2. . Reconstructions of c_1 and c_2 . The results for c_1 are denoted by black dots and the results for c_2 are denoted by gray dots. $\widetilde{c}_1$ is denoted by solid line and $\widetilde{c}_2$ by a dashed line. The relative error δ_{error} for $c_1 = 0.1443$ in Ω and for $c_2 = 0.0802$ in Ω.

Appendix A. **.** We denote the formal heat operator by $\mathcal{L}(c_j)q = \partial_t q + \mathcal{A}(c_j)q$, with $\mathcal{A}(c_j) = -\partial_x(c_j\partial_x\cdot)$, and its formal adjoint by $\mathcal{L}_j^* = -\partial_t + \mathcal{A}(c_j)$. We set $D(A_j) = \{u \in H_0^1(\Omega);\ \mathcal{A}(c_j)u \in L^2(\Omega)\}$ and $D(L_j^{(*)}) = \{z :\ z \in L^2(0,T;H_0^1(\Omega)),\ \mathcal{L}_j^{(*)}z \in L^2(Q)\}$.

A.1. **Proofs of Theorem 2.3 and Corollary 3.** For simplicity we only consider the system (7). The Carleman estimate (4) applied for (7) with q replaced by q_j and f by f_j gives:

$$\int_Q \Big(\Pi_{-1}|\partial_t q_j|^2 + \Pi_{-1}|A(c_j)q_j|^2 + \Pi_1|\partial_x q_j|^2 + \Pi_3|q_j|^2\Big)\,\mathrm{d}x\,\mathrm{d}t \le$$
$$C\Big(\int_Q \Pi_3|q_j|^2\,\mathrm{d}x\,\mathrm{d}t + \int_Q \Big(\Pi_0\sum_{k=1}^J(|\partial_x q_k|^2 + |q_k|^2) + \Pi_0|f_j|^2\Big)\,\mathrm{d}x\,\mathrm{d}t\Big).$$

We then sum these inequalities over $j = 1,\dots,J$. Thanks to $CJ\Pi_0 \le \Pi_1/2$, we obtain (9).

Now we replace q_2 by $a_{12}^{-1}(L_1q_1 - \mathbf{b}_{11}\partial_x q_1 - a_{11}q_1 - f_1)$ in Q. Since $\overline{\omega}\cap S = \emptyset$ we can apply the same method than in [8]. Hence (10).

A.2. **Proof of Corollary 2.** Set $\mathcal{L}(\xi)q := \partial_t q - \partial_x(\xi\partial_x q)$, $\mathcal{P}q := \mathcal{L}(\xi)q + b\cdot\partial_x q + aq$, and fix q solution of (2) with $f = f_0 + \partial_x f_1$. By definition, we have, for any $g \in L^2(Q)$,

$$\int_Q qg\,\mathrm{d}x\,\mathrm{d}t = \int_Q (f_0\, z - f_1\partial_x z)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega q_0\, z(0)\,\mathrm{d}x\,, \tag{40}$$

where $z \in D(L^*(\xi))$ is the solution of $P^*z = g$, $z(T) = 0$. Here we have $\mathcal{P}^*z = -\partial_t z - \partial_x(\xi\partial_x z) - \partial_x(bz) + az$, $z|_{\Sigma=0}$. We solve the following equation with unknown $p \in \mathcal{Q} \supset \mathcal{Q}_0 = \mathcal{C}^2([0,T];D(A^2))$, where $\mathcal{Q}$ is defined later:

$$\mathcal{P}^*(\Pi_0\mathcal{P}p) + \Pi_3\chi_\omega p = \Pi_3 q \text{ in } Q. \tag{41}$$

We define on $(\mathcal{Q}_0)^2$ the bilinear symmetric form $m(\cdot,\cdot)$ and, on $\mathcal{Q}_0$, the linear form $b^\natural(\cdot)$ as follows :

$$\begin{aligned} m(p,p') &:= \int_Q \Pi_0\mathcal{P}p\,\mathcal{P}p'\,\mathrm{d}x\,\mathrm{d}t + \int_{(0,T)\times\omega}\Pi_3 pp'\,\mathrm{d}x\,\mathrm{d}t\,,\\ b^\natural(p') &:= \int_Q \Pi_3 qp'\,\mathrm{d}x\,\mathrm{d}t\ \ . \end{aligned}$$

Notice that $\mathcal{Q}_0 \subset D(L)$, and so we can apply the Carleman estimate (4):

$$\int_Q \Pi_3 p^2\,\mathrm{d}x\,\mathrm{d}t \le Cm(p,p)\,, \tag{42}$$

for all $p \in \mathcal{Q}_0$. This implies that $\mathcal{Q}_0$ equipped with $\|p\|_m := \sqrt{m(p,p)}$ is a normed space. Then we denote by $\mathcal{Q}$ the closure of $\mathcal{Q}_0$ with this norm. So $\mathcal{Q}$ equipped with the bilinear form $m(\cdot,\cdot)$ is an Hilbert space and (42) holds for any $p \in \mathcal{Q}$. Thanks to the Cauchy–Schwarz estimate and to (42), we then have for any $p \in \mathcal{Q}$:

$$|b^\natural(p)| \le \Big(\int_Q \Pi_3|q|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac12}\Big(\int_Q \Pi_3|p|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac12} \le C\Big(\int_Q \Pi_3|q|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac12}\|p\|_m\ . \tag{43}$$

So the form $b^\natural$ is m-continuous. Hence, we can apply the Lax–Milgram theorem to the following equation with unknown $p \in \mathcal{Q}$:

$$m(p, p') = b^\natural(p') \text{ for any } p' \in \mathcal{Q}. \tag{44}$$

We denote by p the unique solution in $\mathcal{Q}$ of (44) and we set $w := -\Pi_3 \chi_\omega p$, $g = w + \Pi_3 q$, $z = \Pi_0 \mathcal{P} p$. Notice that $z \in D(L^*(\xi))$, $z(0, \cdot) = z(T, \cdot) = 0$ in Ω, and thanks to (41), we have $P^* z = g = w + \Pi_3 q$. Furthermore, thanks to (40),

$$\int_Q \Pi_3 q^2 \,\mathrm{d}x\,\mathrm{d}t = -\int_Q f_1 \cdot \partial_x z \,\mathrm{d}x\,\mathrm{d}t + \int_Q f_0\, z \,\mathrm{d}x\,\mathrm{d}t + \int_Q \Pi_3 q p \,\mathrm{d}x\,\mathrm{d}t \ .$$

Thanks to (43), we have

$$m(p,p) = b^\natural(p) \le C \left(\int_Q \Pi_3 |q|^2 \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{2}} |m(p,p)|^{\frac{1}{2}} ,$$

and so

$$m(p,p) = \int_Q \Pi_0^{-1} z^2 \,\mathrm{d}x\,\mathrm{d}t + \int_Q (\Pi_3)^{-1} w^2 \,\mathrm{d}x\,\mathrm{d}t \le C \int_Q \Pi_3 |q|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{45}$$

Let us prove the following estimate:

$$\int_Q (\Pi_0)^{-1} z^2 \,\mathrm{d}x\,\mathrm{d}t + \int_Q (\Pi_3)^{-1} w^2 \,\mathrm{d}x\,\mathrm{d}t + \int_Q (\Pi_2)^{-1} |\partial_x z|^2 \,\mathrm{d}x\,\mathrm{d}t \le C \int_Q \Pi_3 q^2 \,\mathrm{d}x\,\mathrm{d}t \ . \tag{46}$$

We set

$$\begin{aligned}
I_1 &:= \int_Q (\Pi_2)^{-1} |\partial_x z|^2 \xi \mathrm{d}x\,\mathrm{d}t, , \\
I_2 &:= -\int_Q (\Pi_2)^{-1} z\, \partial_x(\xi \partial_x z) \,\mathrm{d}x\,\mathrm{d}t \,, \\
I_3 &:= -\int_Q \partial_x(\Pi_2^{-1}) \partial_x z\, z\, \xi \,\mathrm{d}x\,\mathrm{d}t, \\
I_4 &:= \int_Q \Pi_2^{-1} z\, w \,\mathrm{d}x\,\mathrm{d}t \,, \\
I_5 &:= \int_Q \Pi_2^{-1} \Pi_3 z\, q \,\mathrm{d}x\,\mathrm{d}t, \\
I_6 &:= \int_Q \Pi_2^{-1} z\, \partial_t z \,\mathrm{d}x\,\mathrm{d}t \,, \\
I_7 &:= \int_Q \Pi_2^{-1} z\, \partial_x(zb) \,\mathrm{d}x\,\mathrm{d}t - \int_Q \Pi_2^{-1} m z^2 \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

The Green formula holds in Q with the weight $\mathrm{d}x\mathrm{d}t$ for the product $(\Pi_2)^{-1}\, z\, \partial_x(\xi \partial_x z)$, since $\partial_x(\xi \partial_x z) \in L^2(Q)$ and $\Pi_2 \in L^2(H^1)$. Thus $I_1 = I_2 + I_3$. Moreover, since

$$-\partial_x(\xi \partial_x z) = w + \Pi_3 q + \partial_t z + \partial_x(zb) - mz,$$

then

$$I_1 = I_3 + I_4 + I_5 + I_6 + I_7. \tag{47}$$

The estimates on I_k for $3 \le k \le 6$ are the same than in [19, A.2.1] (with similar but not exactly the same notations), and so are already proved. Thanks to

$$\begin{aligned}\Pi_3\,\Pi_0 \le C(\Pi_2)^2 \ \text{ (or equivalently: } (\Pi_2)^{-1} \le C(\Pi_3\,\Pi_0)^{-\frac12}),\\ \Pi_3^2\,\Pi_0 \le C(\Pi_2)^3\,,\\ |\partial_x \Pi_2| \le C\Pi_3\,,\\ |\partial_t \Pi_2| \le Cs^{-1}(\Pi_0)^{-1}(\Pi_2)^2\,,\end{aligned}$$

where $C = C(T,\Omega,\omega,\varphi)$ does not depend on s, we obtain

$$|I_3| \ \le\ \frac12 I_1 + C\int_Q \Pi_3\,|q|^2\,\mathrm{d}x\,\mathrm{d}t\,, \tag{48}$$

$$|I_4|+|I_5|+|I_6| \ \le\ C\int_Q \Pi_3 q^2\,\mathrm{d}x\,\mathrm{d}t\ . \tag{49}$$

The new term is I_7. We have

$$I_7 = -\int_Q \Pi_2^{-1} z\mathbf{b}\partial_x z\,\mathrm{d}x\,\mathrm{d}t - \int_Q (\mathbf{b}\partial_x(\Pi_2^{-1}) + m\Pi_2^{-1})z^2\,\mathrm{d}x\,\mathrm{d}t := I_{71} + I_{72}\ .$$

We can estimate I_{71} as for I_3. Thanks to the Minkovski inequality and to (45), we have

$$\begin{aligned}|I_{71}| \ &\le\ Cs^{-1}\int_Q \Pi_2^{-1}|\partial_x z|^2\,\mathrm{d}x\,\mathrm{d}t + Cs\int_Q \Pi_2^{-1}|z|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\le\ Cs^{-1}I_1 + Cs^{-1}\int_Q \Pi_0^{-1}|z|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\le\ \frac12 I_1 + Cs^{-1}\int_Q \Pi_3|q|^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}\tag{50}$$

We also have

$$\begin{aligned}|I_{72}| \ &\le\ C\int_Q (|\partial_x(\Pi_2^{-1})| + \Pi_2^{-1})|z|^2\,\mathrm{d}x\,\mathrm{d}t \le C\int_Q \Pi_0^{-1}|z|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\le\ C\int_Q \Pi_3|q|^2\,\mathrm{d}x\,\mathrm{d}t\ .\end{aligned}\tag{51}$$

From (47), (48), (49), (50) and (51), we deduce that $|I_1| \le C\int_Q \Pi_3 q^2\,\mathrm{d}x\,\mathrm{d}t$. Hence (46) holds. The conclusion follows, as in [19].

A.3. **Proof of Theorem 2.4.** The system (7) with data $f_j = f_{0j} + \partial_x f_{1j}$, $j = 1,2 \in L^2(H^{-1})$ can be written as

$$\mathcal{P}_j q_j = F_{0j} + \partial_x \mathbf{F}_j \text{ in } Q,$$

where we set $\mathcal{P}_j q_j := \mathcal{L}(c_j)q_j - \mathbf{b}_{jj}\partial_x q_j - a_{jj}q_j$ and

$$\begin{aligned}F_{0j} \ &:=\ \sum_{k\neq j}(-\partial_x \mathbf{b}_{jk} + a_{jk})q_k + f_{0j},\\ \mathbf{F}_j \ &:=\ \sum_{k\neq j} q_k \mathbf{b}_{jk} + \mathbf{f}_j.\end{aligned}$$

For each $j = 1, 2$, we apply Corollary 2, with $(f_0, \mathbf{f})$ replaced by $(F_{0,j}, \mathbf{F}_j)$. Hence we have

$$\begin{aligned}\int_Q \Pi_3 |q_j|^2 \,\mathrm{d}x\,\mathrm{d}t \quad &\leq \quad C\Big(\int_Q \Pi_3 |q_j|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_Q \Pi_0 (|\mathbf{q}|^2 + |f_{0j}|^2) \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_Q \Pi_2 (|\mathbf{q}|^2 + |\mathbf{f}_j|^2) \,\mathrm{d}x\,\mathrm{d}t\Big),\ \forall j = 1, 2,\end{aligned} \tag{52}$$

where $\mathbf{q} := (q_1, q_2)$. We choose s_0 such that

$$2(\Pi_0 + \Pi_2) \leq \frac{1}{2}\Pi_3,\ \forall s > s_0.$$

We then obtain (11) by summing (52) for $1 \leq j \leq 2$.

A.4. **Proof of Corollary 4.** By using (10), we mimick the proof of Corollary 2, with the following minor modifications. Set $\mathbf{P}\mathbf{q} = (P_1 q_1, P_2 q_2)$, $\mathbf{F}_0 = (F_{01}, F_{02})$, $\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2)$, and replace a, $\mathbf{b}$ respectively by $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2)$. Replace also (41) by

$$\mathcal{P}^*(\Pi_0 \mathcal{P}\mathbf{p}) + \Pi_7 \chi_\omega p_2 = \Pi_3 \mathbf{q} \text{ in } Q,$$

and the definition of m and $b^\natural$, w, g, z, respectively by

$$\begin{aligned} m(\mathbf{p}, \mathbf{p}') \quad &= \quad \int_Q \Pi_0 \mathcal{P}\mathbf{p}\,\mathcal{P}\mathbf{p}' \,\mathrm{d}x\,\mathrm{d}t + \int_{(0,T)\times\omega_0} \Pi_7 p_2 p_2' \,\mathrm{d}x\,\mathrm{d}t, \quad b^\natural(\mathbf{p}') = \int_Q \Pi_3 \mathbf{q}\mathbf{p}' \,\mathrm{d}x\,\mathrm{d}t\,, \\ \mathbf{u} \quad &= \quad (0, -\Pi_7 \chi_\omega p_2), \quad \mathbf{g} := \mathbf{u} + \Pi_3 \mathbf{q}, \quad \mathbf{z} = \mathbf{P}^{*\,-1}\mathbf{z}.\end{aligned}$$

We then obtain (12)

A.5. **Proof of lemma 2.5.** We have

$$-\partial_x(\xi \partial_x w) = f,\ -\partial_x(\tilde{\xi} \partial_x \tilde{w}) = \tilde{f}.$$

Define $q_s \equiv \mathrm{e}^{s'\varphi} q$, $s' = s\eta(\theta)$ and

$$P\,w \quad := \quad -\partial_x(\xi_s \partial_x w) + s' \partial_x \varphi\, \xi_s \cdot \partial_x w, \tag{53}$$

$$\tilde{P}\,\tilde{w} \quad := \quad -\partial_x(\tilde{\xi}_s \partial_x \tilde{w}) + s' \partial_x \varphi\, \tilde{\xi}_s \cdot \partial_x \tilde{w}. \tag{54}$$

Set $q = \xi \partial_x w - \tilde{\xi} \partial_x \tilde{w}$, we have $q, q_s \in W^{1,\infty}(\Omega)$. We get

$$\begin{aligned}\int_\Omega q_s^2 \,\mathrm{d}x \quad &\leq \quad \|\partial_x \tilde{u}\|_{L^\infty}^2 \int_\Omega \gamma_s^2 \,\mathrm{d}x + {\xi^2}_{s,\,max} \int_\Omega |\partial_x(w - \tilde{w})|^2 \,\mathrm{d}x, \\ &\leq \quad C_1 \int_\Omega \gamma_s^2 \,\mathrm{d}x + C_2(R; s) \int_\Omega |\partial_x(w - \tilde{w})|^2 \,\mathrm{d}x,\end{aligned} \tag{55}$$

where $C_1, C_2 > 0$. We calculate

$$\begin{aligned} I \quad &:= \quad \int_\Omega (Pw - P'\tilde{w})^2 \rho \,\mathrm{d}x = \int_\Omega (-\partial_x q_s + s' \partial_x \varphi\, q_s)^2 \rho \,\mathrm{d}x \\ &\geq \quad -2s' \int_\Omega \partial_x q_s\, \rho \partial_x \varphi\, q_s \,\mathrm{d}x + {s'}^2 \int_\Omega (\partial_x \varphi\, q_s)^2 \rho \,\mathrm{d}x \equiv s' I_1 + {s'}^2 I_2.\end{aligned}$$

Using $\partial_{\mathbf{n}(x)} \varphi < 0$ on $\partial\Omega$ and thanks to $\rho \partial_x \varphi \in W^{1,\infty}(\Omega)$, an integration by parts gives:

$$\begin{aligned} I_1 \quad &= \quad -\int_\Omega \rho \partial_x \varphi \partial_x (q_s^2) \,\mathrm{d}x = -\sum_{\partial\Omega} \rho \partial_{\mathbf{n}(x)} \varphi q_s^2 + \int_\Omega \partial_x(\rho \partial_x \varphi)\, q_s^2 \,\mathrm{d}x \\ &\geq \quad -C \int_\Omega q_s^2 \,\mathrm{d}x,\end{aligned}$$

for some $C > 0$. Thus, thanks to (55),

$$I_1 \quad \geq \quad -C_1 \int_\Omega \gamma_s^2 \,\mathrm{d}x - C_2(R;s) \int_\Omega |\partial_x(w - \tilde{w})|^2 \,\mathrm{d}x.$$

Moreover, we have, thanks to (**HS1**),

$$\begin{aligned} I_2 \quad &\geq \quad \int_{\Omega\setminus\omega_0} (\partial_x\varphi\, q_s)^2 \rho \,\mathrm{d}x = \int_{\Omega\setminus\omega_0} (\gamma_s \partial_x\varphi\, \partial_x\tilde{w} + c_s\partial_x\varphi\partial_x(w-\tilde{w}))^2\rho\,\mathrm{d}x \\ &\geq \quad C_3 \int_{\Omega\setminus\omega_0} \gamma_s^2 \,\mathrm{d}x - C_4(R;s)\int_\Omega |\partial_x(w-\tilde{w})|^2\,\mathrm{d}x, \end{aligned}$$

for some $C_3, C_4(R;s) > 0$. Hence, there exist positive constants C_1, $C_2(R;s) > 0$, $C_3 > 0$ (depending on $\tilde{\xi}$) such that

$$I \geq C_3 {s'}^2 \int_{\Omega\setminus\omega_0} \gamma_s^2 \,\mathrm{d}x - C_1 s' \int_\Omega \gamma_s^2\,\mathrm{d}x - C_2(s,R)\int_\Omega |\partial_x(w-\tilde{w})|^2\,\mathrm{d}x. \tag{56}$$

But since $Pw - \tilde{P}\tilde{w} = f_s - \tilde{f}_s$, then

$$I \quad \leq \quad C\|f_s - \tilde{f}_s\|^2_{L^2(\Omega)}. \tag{57}$$

Thanks to (56) and (57), we obtain (14).

A.6. **Proof of lemma 2.8.** For the sake of simplicity, we assume that $\theta = T/2$. An integration by parts gives

$$\begin{aligned} B_{j1} \quad &= \quad \int_{\Omega_\theta} \frac{1}{2}\frac{\partial}{\partial t}(\Pi_1(t,x;s)|z_j|^2)\,\mathrm{d}x \\ &= \quad \int_{\Omega_\theta} s\varphi\partial_t\eta\,\Pi_1\,|z_j|^2\,\mathrm{d}x + \int_{\Omega_\theta} \Pi_1 z_j\,\partial_t z_j\,\mathrm{d}x \equiv B_{j2} + B_{j3}. \end{aligned}$$

Upper bounds of B_{j1}. We have $|B_{j2}| \leq CB'_{j2}$ with

$$B'_{j2} \quad := \quad \int_Q s\eta^2\Pi_1\,|z_j|^2\,\mathrm{d}x\,\mathrm{d}t = \int_Q s^{-1}\Pi_3\,|z_j|^2\,\mathrm{d}x\,\mathrm{d}t \quad .$$

Thanks to the Carleman estimate (12) applied to (28), we have:

$$s\,B'_{j2} \leq C(B_4 + \int_Q \Pi_7|z_2|^2\,\mathrm{d}x\,\mathrm{d}t),$$

where we set

$$B_4 \quad := \quad \int_Q \Pi_2|\partial_x\partial_t\mathbf{u}|^2|\overline{\gamma}|^2\,\mathrm{d}x\,\mathrm{d}t,$$

with $\overline{\gamma} = (c_1 - \widetilde{c}_1, c_2 - \widetilde{c}_2)$. For $k \in \mathbb{N}$ we write:

$$\eta^k(t)\mathrm{e}^{2s\eta(t)\varphi} = \eta^k(\theta)\mathrm{e}^{2s\eta(\theta)\varphi}\zeta^k(t)\mathrm{e}^{2s(\zeta(t)-1)\varphi}\,,$$

where

$$\zeta(t) := \frac{\eta(t)}{\eta(\theta)} = \theta(T-\theta)t^{-1}(T-t)^{-1} \geq 1.$$

Hence

$$\Pi_k(t,x;s) \leq \lambda_k\eta^k(\theta)\pi_k(x;s)\,,$$

where we fix $\lambda_k = \lambda_k(s'_0) \in \mathbb{R}$ independently to $s' \geq s'_0 > 0$ so that

$$\sup_{r\geq 1}(r^k\mathrm{e}^{-2s'(r-1)}) \leq \lambda_k.$$

Since $\mathbf{c} \in \mathcal{E}(R)^2$ and thanks to **(H4)**, we obtain

$$\begin{aligned} B_4 &\leq \lambda_2 \int_\Omega \pi_2 |\overline{\gamma}|^2 (\int_0^T |\partial_t \partial_x \mathbf{u}|^2 \,\mathrm{d}t) \,\mathrm{d}x \\ &\leq C(R) \int_\Omega \pi_2 |\overline{\gamma}|^2 \,\mathrm{d}x \leq C(R) s \int_\Omega \pi_1 |\overline{\gamma}|^2 \,\mathrm{d}x. \end{aligned}$$

Thus we obtain

$$\begin{aligned} |B_{j2}| &\leq C B'_{j2} \\ &\leq C(R) \int_\Omega \pi_1 |\overline{\gamma}|^2 \,\mathrm{d}x + C(R;s) \int_Q |z_2|^2 \,\mathrm{d}x \,\mathrm{d}t. \end{aligned} \tag{58}$$

Thanks to Minkovski's estimate, we have $|B_{j3}| \leq B'_{j2} + B_{j5}$ with

$$B_{j5} := \int_Q \eta^{-1} \Pi_0 |\partial_t z_j|^2 \,\mathrm{d}x \,\mathrm{d}t \ .$$

Notice that $\eta(t) \geq \eta(T/2) > 0$. Hence we have

$$C^{-1} B_{j5} \leq \int_Q s^{-3} \Pi_3 |w_j|^2 \,\mathrm{d}x \,\mathrm{d}t \ .$$

Thanks to **(H4)**, an upper bound for B_{j5} is obtained as for B'_{j2} by using the Carleman estimate (12) to (28). We thus have

$$|B_{j5}| \leq C(R) s^{-2} \int_\Omega \pi_1 |\overline{\gamma}|^2 \,\mathrm{d}x + C(R;s) \int_Q |w_2|^2 \,\mathrm{d}x \,\mathrm{d}t. \tag{59}$$

So from (58), (59) we obtain (30).

Lower bounds for B_1. Thanks to (28), (29), (**H1**), (**H2**), (**H3**) and by applying Proposition 2, we obtain

$$\begin{aligned} \int_\Omega \pi_3 |c_j - \widetilde{c}_j|^2 \,\mathrm{d}x &\leq C_1 \Big(\int_{\Omega \setminus \omega} \pi_1 |f_j - \tilde{f}_j|^2 \,\mathrm{d}x + \int_\omega \pi_2 |f_j - \tilde{f}_j|^2 \,\mathrm{d}x \Big) \\ &\quad + C_2(R;s) (\|v_j - \tilde{v}_j\|^2_{H^1(\Omega)} + \|v_j - \tilde{v}_j\|^2_{H^2(\omega)} \Big) \\ &\leq C'_1 \Big(\int_{\Omega \setminus \omega} \pi_1 |z_j(\theta)|^2 \,\mathrm{d}x + \int_\omega \pi_2 |z_j(\theta)|^2 \,\mathrm{d}x + \int_{\Omega \setminus \omega} \pi_1 |\mathbf{v} - \tilde{\mathbf{v}}|^2 \,\mathrm{d}x \\ &\quad + \int_\omega \pi_2 |\mathbf{v} - \tilde{\mathbf{v}}|^2 \,\mathrm{d}x \Big) + C_2(R;s) (\|v_j - \tilde{v}_j\|^2_{H^1(\Omega)} + \|v_j - \tilde{v}_j\|^2_{H^2(\omega)} \Big) \\ &\leq C B_1 + C'_1 \int_\omega \pi_2 |\mathbf{z}(\theta)|^2 \,\mathrm{d}x + C_2(R;s) \Big(\|\mathbf{v} - \tilde{\mathbf{v}}\|^2_{H^1(\Omega)} \\ &\quad + \|\mathbf{v} - \tilde{\mathbf{v}}\|^2_{H^2(\omega)} \Big), \end{aligned}$$

for all $s \geq s_0$. Hence (31).

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